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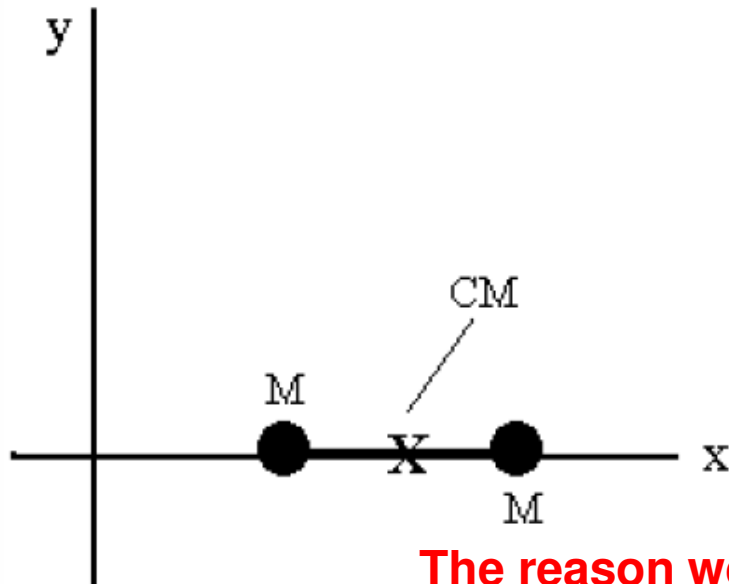
# Center of Mass

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AP Physics C

# Momentum of point masses

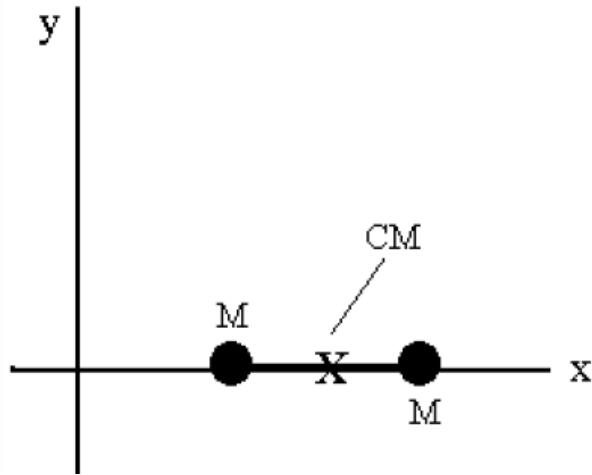
Up to the point in the course we have treated everything as a single point in space no matter how much mass it has. We call this "Special point" the **CENTER OF MASS**. The **CM** is basically a coordinate which represents the average position of mass.



Consider an extended rigid body made up of 2 or more point masses, such as a dumb-bell. We want to treat this as a system, but we don't want to just assume that the CM is in the middle (even though we know it is).

**The reason we are concerned here is because if this system were to move, how would you determine the momentum? Would you just assume ALL the mass was in the middle and treat it as a point?**

# Center of Mass



$$X_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \rightarrow \frac{mx_1 + mx_2}{2m}$$

$$X_{CM} = \frac{x_1 + x_2}{2}$$

Let's assume that the rod that holds the two masses together is very light and that its mass is negligible compared to the weighted ends.

To calculate the "X" position of the center of mass we would do the following, which looks like taking an average.

The formal equation to find the **CENTER OF MASS!** Keep in mind that you can also use this for the Y and Z coordinate!

$$X_{CM} = \frac{\sum_{i=1}^N m_i x_i}{M_{total}}$$

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## Example

Consider the following masses and their coordinates which make up a "discrete mass" rigid body.

$$m_1 = 5.0\text{kg}, \quad \vec{r}_1 = 3\hat{i} - 2\hat{k}$$

$$m_2 = 10.0\text{kg}, \quad \vec{r}_2 = -4\hat{i} + 2\hat{j} + 7\hat{k}$$

$$m_3 = 1.0\text{kg}, \quad \vec{r}_3 = 10\hat{i} - 17\hat{j} + 10\hat{k}$$

What are the coordinates for the center of mass of this system?

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## Example cont'

$$\vec{r}_{cm_x} = \frac{\sum_{i=1}^N m_i x_i}{M} \quad \vec{r}_{cm_y} = \frac{\sum_{i=1}^N m_i y_i}{M} \quad \vec{r}_{cm_z} = \frac{\sum_{i=1}^N m_i z_i}{M}$$

$$m_1 = 5.0\text{kg}, \quad \vec{r}_1 = 3\hat{i} - 2\hat{k}$$

$$m_2 = 10.0\text{kg}, \quad \vec{r}_2 = -4\hat{i} + 2\hat{j} + 7\hat{k}$$

$$m_3 = 1.0\text{kg}, \quad \vec{r}_3 = 10\hat{i} - 17\hat{j} + 10\hat{k}$$

$$\vec{r}_{cm_x} = \frac{\sum_{i=1}^N m_i x_i}{M} = \frac{(5)(3) + (10)(-4) + (1)(10)}{16} = \mathbf{-0.94}$$

$$\vec{r}_{cm_y} = \frac{\sum_{i=1}^N m_i y_i}{M} = \frac{(5)(0) + (10)(2) + (1)(-17)}{16} = \mathbf{0.19}$$

$$\vec{r}_{cm_z} = \frac{\sum_{i=1}^N m_i z_i}{M} = \frac{(5)(-2) + (10)(7) + (1)(10)}{16} = \mathbf{4.4}$$

$$\vec{r}_{cm} = -0.94\hat{i} + 0.19\hat{j} + 4.4\hat{k}$$

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## A continuous body?

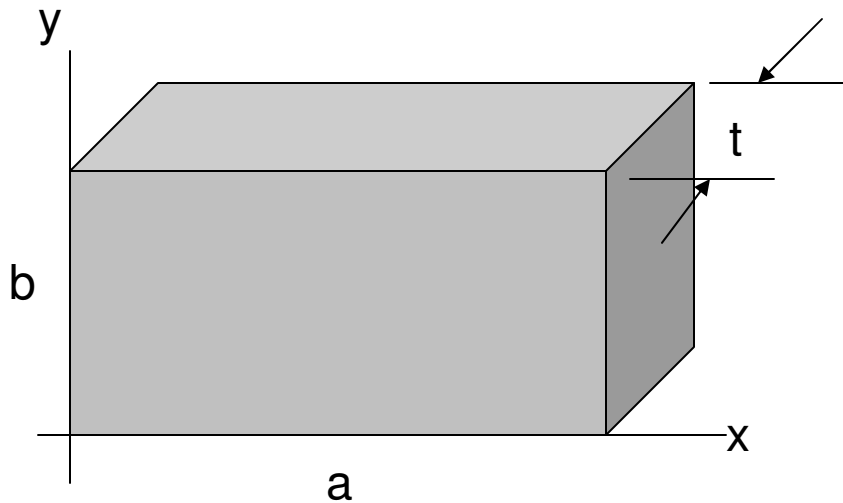
What if you had a body that had a non-uniform mass distribution throughout its structure? Since for a rigid body we used SUMMATION,  $\Sigma$ , it only make sense that we would use INTEGRATION to sum all of the small individual masses thus allowing us to determine the center of mass of an object.

$$X_{CM} = \frac{\int x dm}{\int dm} \rightarrow \frac{1}{M} \int x dm$$

**Here is the SAME equation as before yet with the appropriate calculus symbols we need. The equation is the same for the y, and z positions.**



# Example



Consider a rectangular solid of width "a" , height "b" and thickness "t", and whose whole mass is M. Determine the X coordinate for the center of mass

So to solve practical problems we need to define a set of spatial expressions

$$\lambda = \frac{M}{L} \quad \sigma = \frac{M}{A} \quad \rho = \frac{M}{V}$$

Linear Mass Density

Surface Area Density

Volume Density

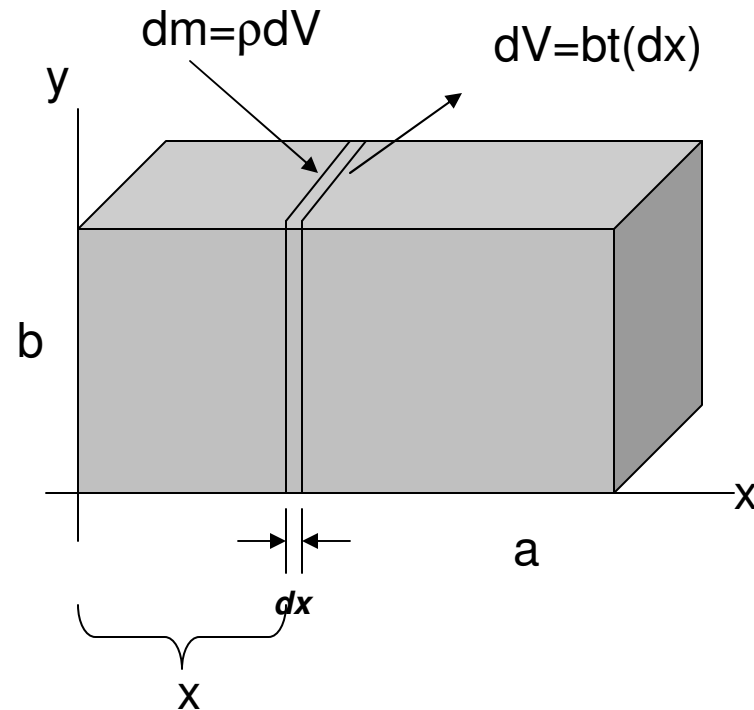
# Macro and Micro

To be able to problems like this it is useful to look at the object **MACROSCOPICALLY** and **MICROSCOPICALLY**. The “micro” part makes sense as zooming in to sum small parts is a fundamental theme in calculus. In this particular situation, volume density will serve our needs,

$$\text{Macro} \rightarrow \rho = \frac{\text{mass}}{\text{volume}} = \frac{M}{abt}$$

$$\text{Micro} \rightarrow \rho = \frac{dm}{dV} = \frac{dm}{bt dx}$$

$$dm = \rho bt(dx)$$



# Example Cont'

$$X_{cm} = \frac{\int x dm}{\int dm} \rightarrow \frac{1}{M} \int x dm$$

$$\text{Macro} \rightarrow \rho = \frac{\text{mass}}{\text{volume}} = \frac{M}{abt}$$

$$\text{Micro} \rightarrow \rho = \frac{dm}{dV} = \frac{dm}{btdx}$$

$$dm = \rho bt(dx)$$

You can't integrate "x" with respect to mass. But we do know an alternate expression for "dm"

$$X_{cm} = \frac{1}{M} \int_{x=0}^a x dm \rightarrow \frac{1}{M} \int_{x=0}^a x(\rho bt) dx$$

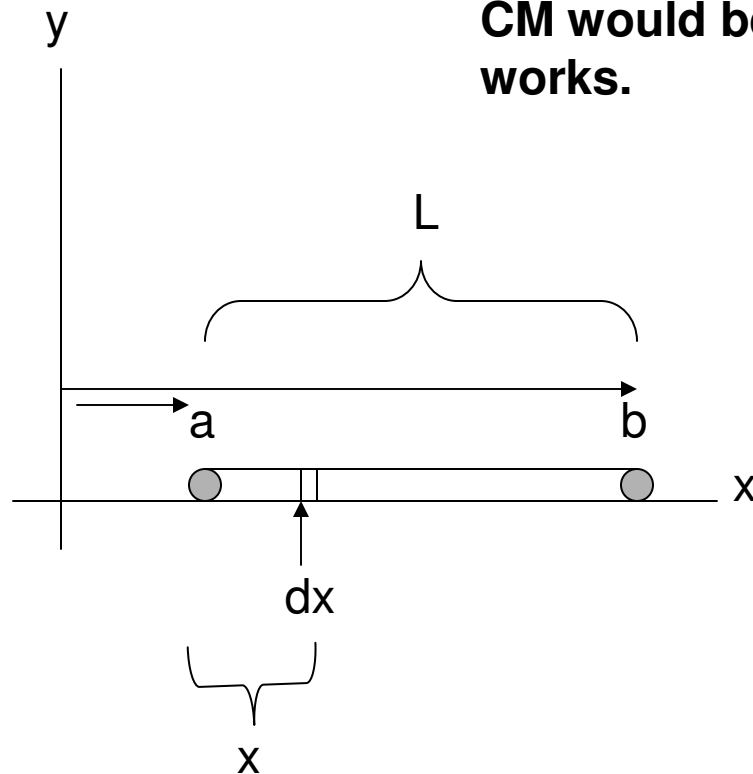
$$X_{cm} = \frac{\rho bt}{M} \int_{x=0}^a x dx$$

$$X_{cm} = \frac{\rho bt}{M} \left| \frac{x^2}{2} \right|_{x=0}^a \rightarrow \frac{\rho bt}{M} \left( \frac{a^2}{2} - 0 \right), \quad \rho = \frac{M}{abt}$$

$$X_{cm} = \frac{1}{a} \left( \frac{a^2}{2} - 0 \right) = \mathbf{a/2}$$

# Example

A 1-D thin **uniform** rod is aligned along the x-axis between  $x = a$  &  $x = b$ . The total mass of the ROD is  $M$ . Determine the X coordinate for the center of mass. We certainly can guess that the CM would be  $(a+b)/2$ , but let's see if our method works.



$$\text{Macro} \rightarrow \lambda = \frac{M}{L} = \frac{M}{b-a}$$

$$\text{Micro} \rightarrow \lambda = \frac{dm}{dx}$$

$$dm = \lambda dx$$

# Example

$$X_{CM} = \frac{\int x dm}{\int dm} \rightarrow \frac{1}{M} \int x dm$$

$$\text{Macro} \rightarrow \lambda = \frac{M}{L} = \frac{M}{b-a}$$

$$\text{Micro} \rightarrow \lambda = \frac{dm}{dx}$$

$$dm = \lambda dx$$

$$X_{cm} = \frac{1}{M} \int_a^b x dm \rightarrow \frac{1}{M} \int_a^b x(\lambda) dx$$

$$X_{cm} = \frac{\lambda}{M} \int_a^b x dx$$

$$X_{cm} = \frac{\lambda}{M} \int_a^b x dx$$

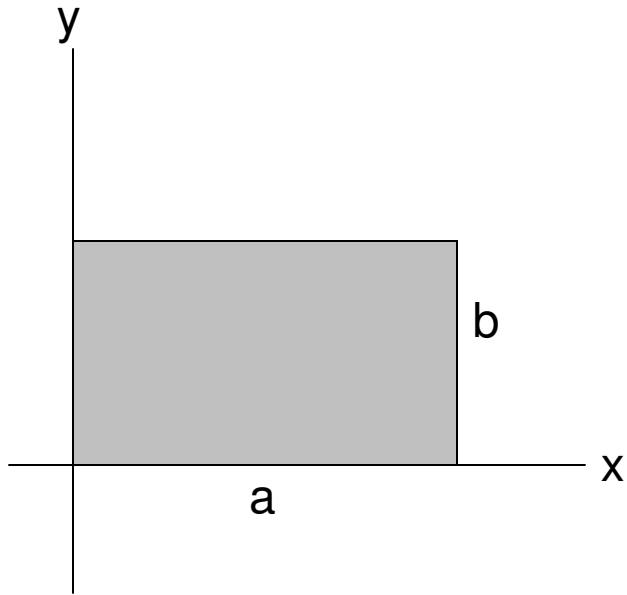
$$X_{cm} = \frac{\lambda}{M} \left[ \frac{x^2}{2} \right]_a^b \rightarrow \frac{\lambda}{M} \left( \frac{b^2}{2} - \frac{a^2}{2} \right), \quad \lambda = \frac{M}{b-a}$$

$$X_{cm} = \frac{1}{b-a} \left( \frac{b^2}{2} - \frac{a^2}{2} \right) \rightarrow \frac{b^2 - a^2}{2(b-a)} \rightarrow \frac{(b+a)(b-a)}{2(b-a)}$$

$$X_{cm} = \frac{b+a}{2}$$

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# Your turn!



A uniform *plate* of mass  $M$  is situated on an  $x$ - $y$  axis. Determine the  $x$  coordinate of its center of mass.

